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# Periodic motions in systems with viscous friction

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**Abstract:** An autonomous dissipative system with one degree of freedom in the presence of a small parameter is considered. It is proved that for sufficiently small values of the parameter in such a system there is a unique limit cycle, it is stable, its period smoothly depends on the small parameter. We apply this theory to several certain systems. The problem of motion of a rigid body (tripod or monopod) freely rotating around a fixed vertical axis and leaning on a horizontal plane with isotropic or anisotropic linear viscous friction uniformly rotating around a fixed vertical axis is considered. If the distance between the axes of rotation of the supporting plane and the rigid body is small enough, then the rigid body asymptotically goes into the unique periodic mode of motion. For both models of friction the dependence of the period of the limit cycle on the small distance between the axes of rotation of the plane and the rigid body is explored analytically. An approximate formula connecting this period and the coefficients of friction in the isotropic and anisotropic cases is found. Numerical simulation of the researched systems is carried out and it is shown that the analytically found dependence for the period can be used to determine the parameters of the viscous friction models.

**Keywords:** dissipative system, small parameter, stable limit cycle, viscous friction.

## 1. Introduction

An important problem in the research of systems with friction is to develop methods for determining the parameters of the friction model. The analysis of the special modes of motion of systems with friction can give information about the parameters of the friction model. Such an approach in combination with experimental methods is considered in [1]. We also note the paper [2], in this one the coefficients of dry and viscous friction in the pendulum joint are determined by its oscillation amplitude. In this paper systems with viscous friction in which stable limit cycles occur are considered. The period of the limit cycle depends on the parameters of the system, in particular on the coefficients of viscous friction. Such systems are generalized to a wider class of autonomous systems with one degree of freedom and a small parameter, where the function that defines the right-hand side of the equation has some restrictions.

## 2. Description of the mechanical system and main result

Let us consider a mechanical system with one degree of freedom, the equation of motion of which has the form

$$\ddot{\varphi} = F(\varphi, \dot{\varphi}, \varepsilon), \quad F(\varphi, \omega, 0) = 0, \quad \left.\frac{\partial F}{\partial \dot{\varphi}}\right|_{\dot{\varphi}=\omega, \varepsilon=0} < 0 . \qquad (1)$$

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Here $F(\varphi, \dot{\varphi}, \varepsilon)$ is a smooth (minimum class $C^4$) function, periodic in the variable $\varphi$, $\varepsilon$ is small parameter, $\omega > 0$. Using the Taylor series expansion of the function $F(\varphi, \dot{\varphi}, \varepsilon)$ with respect to the variables $\dot{\varphi}$ and $\varepsilon$ at the point $\dot{\varphi} = \omega, \varepsilon = 0$, we write equation (1) in the form of the system

$$\dot{\varphi} = z, \quad \dot{z} = -h(\varphi, z)(z - \omega) + \varepsilon f(\varphi, z, \varepsilon) . \tag{2}$$

Due to the periodicity in $\varphi$ the phase space of this system is a cylinder. The function $f(\varphi, z, \varepsilon)$ is bounded in a certain band $[0,2\pi] \times [\omega - \delta, \omega + \delta]$ ($\delta > 0$) on the phase cylinder and for small $\varepsilon$, and $h(\varphi, z) > c > 0$ in this band. In mechanical systems the term of the form $-h(\varphi, z)z$ occurs in the presence of viscous friction. Using the Brauer's theorem [3] and the Dulac criterion [4, 5], the following statement is proved.

*Statement.* For sufficiently small $\varepsilon$ in the band $[0,2\pi] \times [\omega - \delta, \omega + \delta]$ on the phase cylinder, where $0 < \delta < \omega$, system (2) has the unique limit cycle, moreover, it is stable and covers the cylinder. The period of this limit cycle is a smooth function of $\varepsilon$.

An example of a mechanical system whose equation of motion has the form (2) is a rigid body (tripod or monopod) that rotates freely around a fixed vertical axis and leans on a uniformly rotating around a fixed vertical axis horizontal plane with isotropic or anisotropic linear viscous friction. In the case of isotropic viscous friction the force $\mathbf{F} = -c\mathbf{v}$ acts on each support point of a rigid body, where $c$ is the coefficient of viscous friction, $\mathbf{v}$ is the speed of the support point relative to the rotating plane. In the case of anisotropic viscous friction the force $\mathbf{F} = -c_1 v^{(1)} \mathbf{e}^{(1)} - c_2 v^{(2)} \mathbf{e}^{(2)}$ acts on each support point, where $c_1, c_2$ are the coefficients of viscous friction, $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}$ are the unit orthogonal vectors that are fixed relative to the rigid body (the support point of the Chaplygin's skate type) or relative to the rotating plane (there are "furrows" on the plane), $v^{(1)} = \langle \mathbf{v}, \mathbf{e}^{(1)} \rangle, v^{(2)} = \langle \mathbf{v}, \mathbf{e}^{(2)} \rangle$. For sufficiently small distance (small parameter) between the axes of rotation of the supporting plane and the rigid body the system has the unique limit cycle, and this cycle is stable. The dependence of the period of this limit cycle on the distance between the axes of rotation of the plane and the rigid body is sought in the form of a series with respect to small parameter. An approximate formula connecting this period and the coefficients of friction in the isotropic and anisotropic cases is found. Numerical simulation of the researched systems is carried out. It is shown that the analytically found dependence for the period can be used to determine the parameters of the viscous friction models.

## 3. Conclusion

The existence of the stable limit cycle for the systems with energy inflow and dissipation and small parameter has been proved. The possibility of determining the parameters of the viscous friction model on the dependence of period of the limit cycle on small parameter has been shown.

## References


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