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Infinite towers in the graph of a Dynamical System

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Abstract: Chaotic attractors, chaotic saddles and periodic orbits are examples of chain-recurrent sets. Using arbitrary small controls, a trajectory starting from any point in a chain-recurrent set can be steered to any other point in that set. The qualitative behaviour of a dynamical system can be encapsulated in a graph whose nodes are the chain-recurrent sets. There is an edge from node A to node B if, using arbitrary small controls, a trajectory starting from any point of A can be steered to any point of B. We discuss physical systems that have infinitely many disjoint coexisting nodes. Such infinite collections can occur for many carefully chosen parameter values. The logistic map is such a system. To illustrate these very common phenomena, we compare the Lorenz system and the logistic map and we show how extremely similar their bifurcation diagrams are in some parameter ranges. Typically, bifurcation diagrams show how attractors change as a parameter is varied. We call ours "graph bifurcation diagrams" to reflect that not only attractors but also unstable periodic orbits and chaotic saddles are shown.

Keywords: graph of a dynamical system, logistic map, Lorenz system, chain-recurrent sets

1. Introduction

The idea of describing the asymptotics of points in a dynamical system through a graph goes back at least to S. Smale [1] that, in 60s, observed that the flow of the gradient vector field ∇f of a generic function f on a compact manifold M can be encoded into a graph Γ whose nodes are the fixed points of f. Γ has an edge between node A and node B if there is an integral trajectory of ∇f asymptotic to A for $t\to-\infty$ and to B for $t\to+\infty$. This idea was generalized by C. Conley [2] in 70s so that it can be applied any kind of continuous or discrete dynamical system. One of his key ideas was replacing the non-wandering set by the larger set of chain-recurrent points. A point x is chain-recurrent if, for any ε>0, there is an ε-controlled loop based at x, where by ε-controlled trajectory we mean a trajectory where, at every integer time, one is allowed to move the point anywhere within an ε radius from its position. In this talk we present our analytical results on the graphs of the logistic map and our numerical results on the Poincare' map of the Lorenz system. Our results suggest that the graph of the logistic map appears as a subgraph in a large set of dissipative higher-dimensional dynamical systems. In particular, many chaotic processes have a much more complicated structure than theoreticians previously expected.

2. Results and Discussion

In [3] we prove that the graph of the logistic map $l_\mu(x)=\mu x(1-x)$ is a tower, namely there is an edge between each pair of nodes. Since in the graph of a dynamical system there cannot be loops, this means that the nodes of the logistic map can be sorted as $N_0, N_1, \dots, N_p$ so that arbitrarily close to N_k there are points asymptoting to $N_{k'}$ for each $k'>k$. Notice that N_p is always the attracting node and that, for 1<μ<4, N_0 is the fixed point 0. The number of nodes p+1 is infinite when the attractor is a Cantor set (the other two possibilities being a periodic orbit and a (chaotic) cycle of intervals). This happens for an uncountable zero-measure set of parameter values.

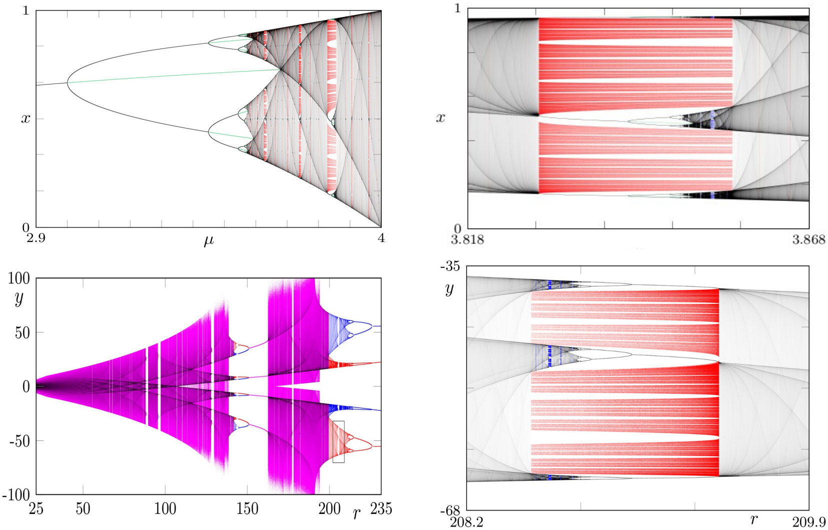

Fig. 1. (Top) Bifurcation diagram of the logistic map. Attractors are painted in shades of gray, repellors in green (periodic orbits), red and blue (Cantor sets). (Bottom) Bifurcation diagram of a Poincare' map of the Lorenz system. The period-3 window shown on the right is the close-up of the one framed in the left picture. Although the systems are completely different, the close-ups of some of their period-3 windows look extremely similar.

Our numerical results, some of which are shown in Fig. 1, suggest that the same graph structure of the logistic map are universal in the sense that they appear in completely unrelated systems such as the celebrated Lorenz system [4].

3. Concluding Remarks

Our numerical investigations suggest that, for some parameter values, even in higher-dimensional dissipative systems infinitely many chain-recurrent sets can arise within a compact set. Notice that this result is “transversal” to the celebrated result of Newhouse that chaotic systems in a compact set can have infinitely many attractors.

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