# Mathematical Modelling of an extended Swinging Atwood Machine

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**Abstract:** An extended model for a variable-length pendulum's mechanical application is being derived from the Swinging Atwood Machine (SAM). An electrical component consisting of an electromagnet and armature coil is attached on the link connected to the counterweight mass on the left-hand side of the modified SAM to provide an excitation force for the system when an electric current is induced. The extended SAM presents a novel SAM concept being derived from a variable-length double pendulum with a suspension between the two pendulums. The equations of motion are simulated to see the trajectory of the two pendulums. The results of original numerical simulations show some compact regions of attraction at some regimes. Therefore, the extended SAM's nonlinear dynamics presented in the current work can be thoroughly studied, and more modifications can be achieved. The new technique can reduce residual vibrations through damping when the desired level of the crane is reached. It can also be used in simple mechatronic and robotic systems.

**Keywords:** variable-length pendulum, swinging Atwood machine, suspension, vibration, damping

## 1. Introduction

Mathematics is useful as a language for characterizing the interaction and relationships among quantifiable concepts, or in mathematical terms, variables. Mathematical modelling has become an increasingly subject as computers expand our ability to translate mathematical equations and formulations into concrete conclusions concerning the natural and artificial world we live in. Furthermore, focusing on mathematical concepts gives the ability to transfer knowledge from one setting to another, which will significantly be enhanced and easy modifications and application for the real-time implementation of engineering projects.

The variable-length pendulum is a physical concept associated with parametric oscillations governed by certain forms of differential equations and functional principles [1, 2]. A parametric oscillator can be treated as a harmonic oscillator whose physical features change over time [3]. Some specific time-dependent variables are associated with the resonance frequency or damping of the oscillator.

Different methods are used in deriving the mathematical equation of engineering systems. The commonly used approach for dynamical systems is the Euler-Lagrange method, Hamiltonian principle, and Newton's second law of motion. The presented mathematical model is derived from the SAM illustrated in [3].

## 2. System description

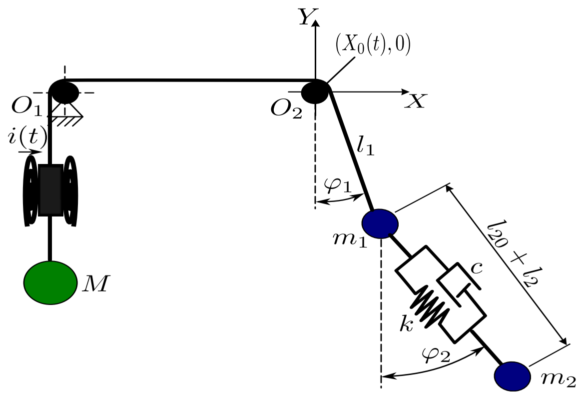


**Fig. 1** Schematic diagram of the proposed modification of the SAM model

Fig. 1 shows the modified SAM, respectively. Another spring pendulum is added on the opposite side of the counter mass $M$. A suspension system with a stiffness $k$ and a damper $c$ placed between the two pendulums with masses $m_1$ and $m_2$. Point $0_1$ is fixed, while $0_2$ is movable and can oscillate in the plane $(Y, X)$, which allow the variation of the length $l_1$ and the double pendulum couplings. An electrical component consisting of an electromagnet and armature coil is attached on the link connected to the counterweight mass on the left-hand side of the modified SAM to provide an excitation force for the system when an electric current is induced. The length $l_{20}$ is measured between the two pendulums, and $l_2$ is the extension due to the spring between the two pendulums.

### *2.1 The equations of motion of the Modified SAM model*

Kinetic Energy:

$$T = \frac{1}{2}M\dot{l}_1^2 + \frac{1}{2}m_1\left(\dot{l}_1^2 + l_1^2 + \dot{\varphi}_1^2\right) - m_2\dot{l}_1 l_2\dot{\varphi}_2 s_{21} - l_{20}m_2\ddot{l}_1\dot{\varphi}_2 s_{21} + m_2 l_1\dot{l}_2\dot{\varphi}_1 s_{21} - m_2 l_1 l_2\dot{\varphi}_1\dot{\varphi}_2 c_{21} + l_{20}m_2 l_1\dot{\varphi}_1\dot{\varphi}_2 c_{21} + m_2\dot{l}_1\dot{l}_2 c_{21} + \frac{1}{2}m_2 l_2^2\dot{\varphi}_2^2 + l_{20}m_2 l_2\dot{\varphi}_2^2 + \frac{1}{2}m_2 l_{20}^2\dot{\varphi}_2^2 + \frac{1}{2}m_2 l_1^2\dot{\varphi}_1^2 + \frac{1}{2}m_2\dot{l}_2^2 + \frac{1}{2}m_2\dot{l}_1^2 \quad (1)$$

Potential Energy:

$$U = \frac{1}{2}kl_2^2 + Mgl_1 - m_1gl_1c_{f1} - m_2g\left(l_1c_{f1} + (l_{20} + l_2)c_{f1}\right) \quad (2)$$

For simplicity for define: $c_{f1} = \cos\varphi_1$, $s_{f1} = \sin\varphi_1$, $c_{f2} = \cos\varphi_2$, $s_{f2} = \sin\varphi_2$, $c_{21} = \cos(\varphi_2 - \varphi_1)$, and $s_{21} = \sin(\varphi_2 - \varphi_1)$.
Where: $\varphi_1$ and $\varphi_2$ are the angles between the movable pulley with length $l_1$ and the angle between the first pendulum with a length $(l_{20} + l_2)$, respectively. With the Lagrange equation, $L = T - U$, we find four degrees of freedom $(l_1, l_2, \varphi_1,$ and $\varphi_2)$. The Euler-Lagrange equation yields:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = 0 \;\; i = 1, 2, 3, 4$ and $R =$ Rayleigh dissipation function:
The following equations are obtained using the Euler-Lagrange equation:
For $q_1 = l_1$:

$$\ddot{l}_1 = \frac{1}{m_1}\left(-\,gm_2s_{f1}s_{21} - gm_2c_{f1}c_{21} + c\dot{l}_2c_{21} + kl_2c_{21} + m_1l_1\dot{\varphi}_1^2 - \ddot{X}_0m_1s_{f1} + gm_2c_{f1} + m_1gc_{f1} - Mg\right) \quad (3)$$

For $q_2 = l_2$:

$$\ddot{l}_2 = \frac{gm_2s_{f1}s_{21}}{m_1} + gm_2s_{f1}s_{21} + \ddot{X}_0c_{f1}s_{21} + \ddot{X}_0s_{f1}c_{21} - \frac{gm_2c_{f1}c_{21}}{m_1} - gc_{f1}c_{21} + \frac{Mgc_{21}}{m_1} + l_2\dot{\varphi}_2^2 + l_{20}\dot{\varphi}_2^2 + \frac{gm_2c_{f2}}{m_1} + gc_{f2} - \frac{c\dot{l}_2}{m_2} - \frac{c\dot{l}_2}{m_1} - \frac{k\dot{l}_2}{m_2} - \frac{k\dot{l}_2}{m_1} \quad (4)$$

For $q_3 = \varphi_1$:

$$\ddot{\varphi}_1 = -\frac{gm_2c_{f1}c_{21}}{m_1l_1} + \frac{c\dot{l}_2s_{21}}{m_1l_1} + \frac{kl_2s_{21}}{m_1l_1} + \frac{gm_2s_{f1}s_{21}}{m_1l_1} - \frac{2\dot{l}_1\dot{\varphi}_1}{l_1} - \frac{gm_2s_{f1}}{m_1l_1} - \frac{gs_{f1}}{l_1} - \frac{\ddot{X}_0c_{f1}}{l_1} \quad (5)$$

For $q_4 = \varphi_2$:

$$\ddot{\varphi}_2 = \frac{1}{m_1l_2 + l_{20}m_1}\left(m_1\ddot{X}_0s_{f1}s_{21} + gm_2c_{f1}s_{21} + gm_1c_{f1}s_{21} - Mgs_{21} + gm_2s_{f1}c_{21} + gm_1s_{f1}c_{21} + m_1\ddot{X}_0c_{f1}c_{21} - 2m_1\dot{l}_2\dot{\varphi}_2 - gm_2s_{f2} - gm_1s_{f2}\right) \quad (6)$$

$\ddot{X}_0 = -\omega^2 f_0 \sin(\omega t)$. $f_0$ is the excitation force, and $\omega$ is the excitation frequency.

## 3. Result and Discussion

Equation (3) – (4) are simulated, and the results are shown in Fig. 2 and 3 for $\varphi_1$, $l_1$ and $\varphi_2$, $(l_{20} + l_2)$, respectively. The results described the modified SAM concepts that demonstrate the chaotic dynamics with a double pendulum and suspension system between the two pendulums excited by an electromagnet and armature. The excitation frequency has an impact on the system. Therefore, only the parameters that show good trajectories are presented.

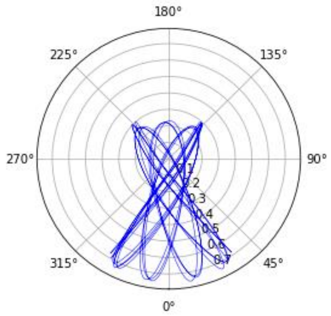


**Fig. 2** An orbit of the Modified SAM $(\varphi_1, l_1)$

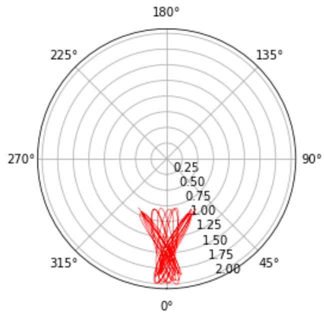


**Fig. 3** An orbit of the Modified SAM $(\varphi_2, (l_{20} + l_2))$

## 4. Conclusion

The presented results show the nonsingular orbit under swinging, with no physical contact between the swinging assemble and the fixed points. Interestingly, in some regimes, compact regions of attraction such as can be seen in Fig. 2 and 3 appear in the system. Therefore, the nonlinear dynamics of the presented modified SAM can be thoroughly studied, and more modification can be achieved.

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