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Pullback and forward dynamics of nonautonomous integrodifference equations: Basic constructions

HUY HUYNH[1*], PETER E. KLOEDEN[2], CHRISTIAN PÖTZSCHE[1]

1. Institut für Mathematik, Universität Klagenfurt, 9020 Klagenfurt, Austria [ORCID: 0000-0002-7364-5702]
2. Mathematisches Institut, Universität Tübingen, 72076 Tübingen, Germany [ORCID: 0000-0001-9505-2257]
* Presenting Author

Abstract: In theoretical ecology, models describing the spatial dispersal and the temporal evolution of species having non-overlapping generations are often based on integrodifference equations. For various such applications the environment has an aperiodic influence on the modelsleading to nonautonomous integrodifference equations. In order to capture their long-term behaviour comprehensively, both pullback and forward attractors, as well as forward limit sets are constructed for general infinite-dimensional nonautonomous dynamical systems indiscrete time. While the theory of pullback attractors, but not their application to integrodifference equations, is meanwhile well-established, the present novel approach is needed inorder to understand their future behaviour.

Keywords: pullback attractor, forward attractor, forward limit set, nonautonomous difference equation, integrodifference equation

1. Introduction

Integrodifference equations (IDEs) occur as temporal discretisations of integrodifferential equations or as time-1-maps of evolutionary differential equations. We are interested in their iterates from a dynamical systems perspective, especially the long term behaviour of recursions based on a fixed nonlinear integral operator. In applications, the iterates for instance represent the spatial distribution of interacting species over a habitat.

2. Results and Discussion

One of the central questions in this paper is the existence and structure of an attractor. These invariant and compact sets attract bounded subsets of an ambient state space and fully capture the asymptotics of an autonomous dynamical system [4]. Extending this situation, the first main part of this paper is devoted to general nonautonomous difference equations in complete metric spaces. In particular, only a combination of several attractor notions yields the full picture:

- *Pullback attractors* [2, 5, 8, 10] are compact, invariant nonautonomous sets which attract all bounded sets from the past. As fixed target problem, they are based on previous information, at a fixed time from increasingly earlier initial times. Consisting of bounded entire solutions to a nonautonomous system [10], a pullback attractor can be seen as an extension of the global attractor to nonautonomous problems and apparently captures the essential dynamics to a certain point. However, pullback attractors reflect the past rather than the future of systems [7].
- *Forward attractors* [8] are also compact and invariant nonautonomous sets. This concept depends on information from the future and given a fixed initial time, the actual time increases beyond all bounds. Forward attractors are not unique, independent of pullback attractors, but

often do not exist. Nevertheless, we will describe forward attractors using a pullback construction, even though this has the disadvantage that information on the system over the entire time axis of integers is required.

- *Forward limit sets* is also a concept related to the information from the future. They correctly describe the asymptotic behaviour of all forward solutions to a nonautonomous difference equation [6]. These limit sets have forward attraction properties, but different from pullback and forward attractors, they are not invariant and constitute a single compact set, rather than a nonautonomous set. Nonetheless, asymptotic forms of positive and negative invariance do hold.

The initial construction of forward attractors and forward limit sets in [6] requires a locally compact state space, but recent continuous-time results in [3], which extend these to infinite-dimensional dynamical systems, will be transferred here.

For the second purpose, the above abstract setting allows concrete applications to a particularly interesting class of infinite-dimensional dynamical systems in discrete time, namely IDEs. We provide sufficient criteria for the existence of the above-mentioned notions of various IDEs. In particular, we illustrate the above theoretical results by studying pullback attractors and forward limit sets.

More complicated equations and the behaviour of attractors under spatial discretisation will be tackled in future papers.

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References

[1] ATKINSON K.E: A survey of numerical methods for solving nonlinear integral equations. *J. Integr. Equ. Appl.* 1992, **4**(1):15-46.

[2] CARVALHO A.N, LANGA J.A, ROBINSON J.C: *Attractors for Infinite-dimensional Non-autonomous Dynamical Systems.* Applied Mathematical Sciences 182. Springer: Berlin, 2012.

[3] CUI H, KLOEDEN P.E, YANG M: Forward omega limit sets of nonautonomous dynamical systems. *Discrete Contin. Dyn. Syst. Ser. B* 2019, **13**:1103-1114.

[4] HALE J.K: *Asymptotic Behaviour of Dissipative Systems.* Mathematical Surveys and Monographs 25. AMS: Providence, 1988.

[5] KLOEDEN P.E: Pullback attractors in nonautonomous difference equations. *J. Differ. Equa. Appl.* 2000, **6**(1):33-52.

[6] KLOEDEN P.E, LORENZ T: Construction of nonautonomous forward attractors. *Proc. Am. Math. Soc.* 2016, **144**(1):259-268.

[7] KLOEDEN P.E, PÖTZSCHE C, RASMUSSEN M: Limitations of pullback attractors for processes. *J. Differ. Equ. Apl.* 2012, **18**(4):693-701.

[8] KLOEDEN P.E, RASMUSSEN M: *Nonautonomous Dynamical Systems*. Mathematical Surveys and Monographs 176. AMS: Providence, 2011.

[9] KOT M, SCHAFFER W.M: Discrete-time growth-dispersal models. *Math. Biosci.* 1986, **80**:109-136.

[10] PÖTZSCHE C: *Geometric Theory of Discrete Nonautonomous Dynamical Systems.* Lecture Notes in Mathematics, Vol. 2002. Springer: Berlin, 2010.