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On Transformations of Closed Invariant Curves in Piecewise-Smooth Maps

ZHANYBAI T. ZHUSUBALIYEV[1*], VIKTOR AVRUTIN[2], FRANK BASTIAN[3]

1. Department of Computer Science, Dynamics of Non-Smooth Systems International Scientific Laboratory, Southwest State University, Russia [0000-0001-5534-9902]
2. Institute for Systems Theory and Automatic Control, University of Stuttgart, Germany [0000-0001-7931-8844]
3. Institute for Systems Theory and Automatic Control, University of Stuttgart, Germany and Department of Applied Mathematics, University College Cork, Ireland [0000-0003-1910-024X]

* Presenting Author

Abstract: In this work we discuss a transformation of a smooth closed invariant curve associated with quasiperiodic dynamics into a piecewise smooth one, and its subsequent transitions to chaos. As an intermediate step, the latter transition involves a transformation of a closed invariant curve into a closed-invariant-curve-like chaotic attractor.

Keywords: closed invariant curve, closed invariant curve-like chaotic attractor, piecewise-smooth maps, expansion bifurcation, border collision

1. Introduction

Many problems in engineering and applied science lead us to consider piecewise-smooth maps. Examples of such systems include power-electronic and pulse-modulated control systems mechanical systems with dry friction or impacts, as well as systems involving thresholds, constrains, and decision-making processes in economics and social sciences. In addition to the bifurcations occurring in smooth systems, piecewise smooth systems demonstrate several further bifurcation phenomena, as, e.g., border collision bifurcations. For historical reasons, most deeply studied are bifurcations involving fixed points and cycles, which the corresponding effects for other kinds of invariant sets are still far away from being understood completely.

Closed invariant curves associated with quasiperiodic dynamics is a specific type of dynamic behavior characterized by two or more oscillatory modes with incommensurable frequencies. It is well-known that in smooth maps this type of dynamics appears via a Neimark–Sacker bifurcation, while in piecewise smooth maps it may also appear via a border collision bifurcation [1].

2. Results

In this work, we investigate a piecewise smooth 2D map introduces initially in [2] (see also [3])

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = F(x_k, y_k), \quad F(x,y) = \begin{cases} F_{\mathcal{L}}(x,y) & \text{if } (x,y) \in \mathcal{I}_{\mathcal{L}}; \\ F_{\mathcal{M}}(x,y) & \text{if } (x,y) \in \mathcal{I}_{\mathcal{M}}; \\ F_{\mathcal{R}}(x,y) & \text{if } (x,y) \in \mathcal{I}_{\mathcal{R}}, \end{cases}$$

where

$$F_{\mathcal{L}}(x,y) = \begin{pmatrix} e^{\lambda_1}x - e^{\lambda_1} + 1 \\ e^{\lambda_2}y - e^{\lambda_2} + 1 \end{pmatrix}; \; F_{\mathcal{M}}(x,y) = \begin{pmatrix} e^{\lambda_1}x - e^{\lambda_1} + e^{\lambda_1(1-z(x,y))} \\ e^{\lambda_2}y - e^{\lambda_2} + e^{\lambda_2(1-z(x,y))} \end{pmatrix}, \; z(x,y) = \frac{\alpha\Gamma}{q}\left(x - \frac{\lambda_1}{\lambda_2}y\right)$$

with

$$\mathcal{I}_{\mathcal{L}} = \{(x,y) \mid y < S_-(x)\}, \; \mathcal{I}_{\mathcal{M}} = \{(x,y) \mid S_-(x) \le y \le S_+(x)\}, \; \mathcal{I}_{\mathcal{R}} = \{(x,y) \mid y > S_+(x)\},$$

$$S_-(x) = \frac{\lambda_2}{\lambda_1}\left(x + \frac{q}{\Gamma} - \frac{q}{\alpha\Gamma}\right), \; S_+(x) = \frac{\lambda_2}{\lambda_1}\left(x + \frac{q}{\Gamma}\right)$$

and report two specific bifurcation scenarios [3]. Here $\lambda_1 = 0.03125 \cdot T_0$, $\lambda_2 = 0.003125 \cdot T_0$ and $q = 13.7278828553996$, $T_0 = 0.9$ s. The parameters α and Γ are control parameters.

In the first scenario, a smooth closed invariant curve associated with quasiperiodic dynamics appears via a Neimark–Sacker bifurcation and undergoes eventually a border-collision. As the dynamics on the closed invariant curve remains unchanged, this scenario can be seen as a kind of persistence border-collision for closed invariant curves. However, as a result of this transition the closed invariant curve becomes piecewise smooth, containing an infinite number of kink points given by the points at which the closed invariant curve intersects the switching manifold, and by the images of these points.

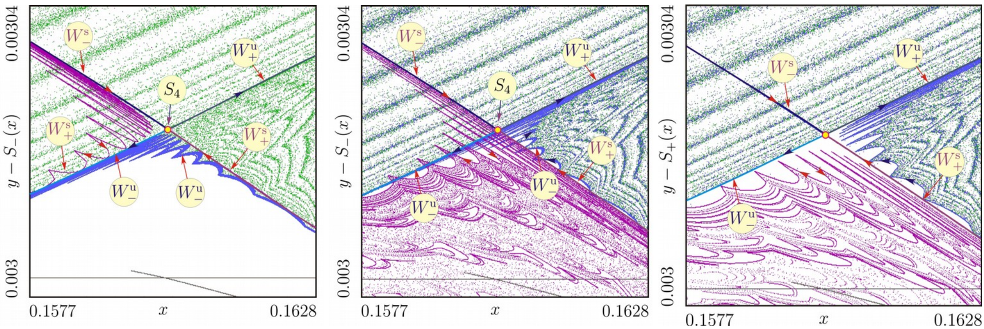

Fig. 1. Homoclinic bifurcation leading to the expansion of a closed-invariant-curve-like chaotic attractor to a large amplitude chaotic attractor. $W^{s}_{\pm}$, $W^{u}_{\pm}$ are the stable and unstable manifoldsof a saddle 4-cycle S_4.

In the second scenario, we consider a transition from an attracting closed invariant curve to a large amplitude chaotic attractor. This transition resembles an expansion bifurcation well-known for chaotic attractors. However, for quasiperiodic attractors no similar bifurcations have been reported so far. In fact, it turns out that the transition takes place in two steps. As a first step, the closed invariant curve undergoes a homoclinic bifurcation and turns into a closed-invariant-curve-like chaotic attractor. Although the overall shape of this attractor very similar to the shape of the previously existing closed invariant curve (which can easily lead to misinterpretations), on a sufficient magnification level the fractal structure of the attractor is recognizable. Eventually, as a second step, the closed-invariant-curve-like chaotic attractor collides with a chaotic repeller and undergoes an expansion bifurcation, leading to the appearance of a large amplitude chaotic attractor.

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