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Codimension-2 bifurcations in a quantum decision making model

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Abstract: We show how a complex bifurcation structure in a 2D parameter space of a piecewise linear discontinuous 2D map modeling the dynamics of a binary choice game using quantum logic can be explained by a few codimension-2 bifurcation points of a type not yet reported in the literature.

Keywords: Border collision bifurcations, Codimension-2 bifurcations, Period adding, Quantum cognition, Binary choices

1. Introduction

Since the influential work of Schelling [1], binary choice games with externalities have become central within social dilemma literature. Previous literature has developed models with agents' behavior mainly based on classical probability. However, recent theoretical and empirical findings have shown how the recurrent contextual effects observed in the psychological literature are better predicted by quantum probability theory [2]. The dynamical population model proposed in [3] introduces quantum cognition in the Schelling model of binary choices (for instance, *left* and *right*) in a minority game. For the sake of simplicity, we assume the population is unitary and variable x is the proportion of population who choose *right*. In order to describe the effect of past choices, the revision protocol of the agents' internal state is given by the context effect rule provided by quantum cognition theory. The phase space $U = [0,1]^2$ is partitioned in four regions:

$$U_L = \{(x_{t-1},x_t) \in U: x_{t-1} > ½,\ x_t > ½\} \qquad U_r = \{(x_{t-1},x_t) \in U: x_{t-1} > ½,\ x_t < ½\}$$
$$U_l = \{(x_{t-1},x_t) \in U: x_{t-1} < ½,\ x_t > ½\} \qquad U_R = \{(x_{t-1},x_t) \in U: x_{t-1} < ½,\ x_t < ½\}$$

2. Results

According to the quantum model, the population dynamics $F : U \to [0,1]$ considers both time periods:

$$F(x_{t-1},x_t) = \begin{cases} f_L(x_{t-1},x_t) = (1-\sin^2\alpha\cos^2\beta)\,x_t & \text{if} \quad (x_{t-1},x_t)\in U_L \\ f_r(x_{t-1},x_t) = (1-\sin^2\alpha\sin^2\beta)\,x_t + \sin^2\alpha\sin^2\beta & \text{if} \quad (x_{t-1},x_t)\in U_r \\ f_l(x_{t-1},x_t) = (1-\cos^2\alpha\sin^2\beta)\,x_t & \text{if} \quad (x_{t-1},x_t)\in U_l \\ f_R(x_{t-1},x_t) = (1-\cos^2\alpha\cos^2\beta)\,x_t + \cos^2\alpha\cos^2\beta & \text{if} \quad (x_{t-1},x_t)\in U_R \end{cases}$$

where: $\alpha\in[0,\pi/2]$ defines the unit vector representing the initial decision makers' beliefs about the two choices; $\beta\in[0,\pi/2]$ the vector representing the revised decision makers' beliefs induced by the previous choice. In the following, values $\alpha,\beta=0$ and $\alpha,\beta=\pi/2$ are referred to as *edges* of the parameter space (although they represent rather symmetry axes than borders). As state variable x_{t-1} appears on the right hand side in the conditions associated with the functions f_l, f_r, f_L, f_R only, the map can also be considered as a piecewise linear bi-valued map with a single border point $x = 1/2$. Functions f_l, f_r, f_L, f_R are contractive and increasing w.r.t. x_t, by construction.

The bifurcation structure in the (α,β) parameter space is illustrated in Fig. 1. It is symmetric w.r.t $\alpha=\pi/4$ and affected by bistability for $\beta<\pi/4$. In the middle of the structure there is a large region associated with a 2-cycle O_{lr} and bounded by curves ξ_{lr} and ξ_{rl} where this cycle undergoes border collision bifurcations (BCBs). The intersection points P_1 and P_2 of these curves with edge $\beta=\pi/2$ represent two organizing centers (two codimension-2 bifurcation points) where two complete period adding structures originate from.

A theory for codimension-2 BCBs in single-valued 1D maps was developed in [4]. In particular, it was shown that at a codimension-2 BCB point –where two different stable cycles collide with the discontinuity from opposite sides– the composite function (defined by the branches of the iterate functions for which the colliding cycles are fixed points) is *continuous*. Then, it was shown that if both involved functions are *contractive* and *locally increasing*, then a complete period adding structure issues from the corresponding codimension-2 BCB point. This theory cannot be directly applied to the points P_1 and P_2, as the edge $\beta=\pi/2$ does not correspond to a BCB. At $\beta=\pi/2$ however, the map has infinitely many fixed points O_L and O_R as f_L, f_R are identity functions. The case is similar to a degenerate+1 bifurcation, although not identical as the fixed points exist neither before nor after $\beta=\pi/2$. Nevertheless, some significant assumptions of the theorem mentioned above are satisfied: at P_1 the composed function defined by branches f_R and $f_r \circ f_l$ is continuous; both branches are increasing; at least branch $f_r \circ f_l$ is contractive. This suggests that, under the present setting, a theorem similar to the one in [4] can be proven: point P_1 must be the origin of a period adding structure with, for example, the regions of complexity level one being associated with cycles $O_{R(lr)^k}$ and $O_{R^k(lr)}$, in agreement with the numerical results.

The same reasoning can be applied to explain the complex bifurcation structure in the lower part of the parameter space. Several regions issuing from P_1 reach the left edge of the parameter space ($\alpha=0$). Then, the codimension-2 points defined by the intersections of one of their boundaries with this edge (at which functions f_l and f_r are identity functions) are the origins of a further complete period adding structure. The regions issuing from these points overlap both pairwise and with the regions issuing from P_1, leading to bistability (see Fig. 1(b)).

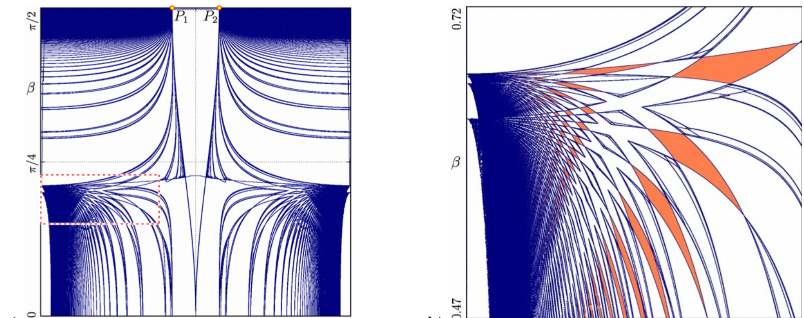

Fig. 1. a) Bifurcation structure in the (α,β) parameter plane. The rectangle in a) is shown magnified in b) where some regions of bistability are highlighted.

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