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Piecewise smooth systems with a pseudo-focus: a normal form approach

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Abstract: The analysis of planar systems with a pseudo-equilibrium point of focus type within its discontinuity line can be more easily done if the classical theory of normal forms is extended for such a case. This methodology allows to remove unessential terms in the expression of the vector field, preserving every point of the discontinuity line and so possible periodic orbits. The approach will be illustrated by considering a rather general family of linear-quadratic planar systems.

Keywords: Discontinuous Systems, Pseudo-Focus, Normal Forms.

1. Introduction

We consider planar piecewise smooth systems with a straight line as the discontinuity manifold. Our main hypothesis is the existence of a pseudo-focus at the origin coming from the collision of an invisible tangency from each side. Our goal is to develop a methodology for the analysis of the dynamics in a neighbourhood of the pseudo-focus, characterizing its stability. In the non-hyperbolic cases, when the pseudo-focus behaves as a weak-focus, we look for the determination of its weakness order, which is associated to the maximum number of limit cycles that can be obtained by perturbation.

The followed approach is based in the obtention of a normal form by making successive near-identity changes of variables and time reparameterizations that must preserve the points of the discontinuity line. The achieved normal form is specially suitable for building the half-return maps, which are the main tools for the analysis.

2. Results and Discussion

Under our hypotheses, we can start from the system

$$\left.\begin{aligned} \dot{x} &= a_{10}^{\pm}x - y + \sum_{p+q\geqslant 2} a_{pq}^{\pm}x^p y^q \\ \dot{y} &= \pm 1 + b_{10}^{\pm}x + b_{01}^{\pm}y + \sum_{p+q\geqslant 2} b_{pq}^{\pm}x^p y^q \end{aligned}\right\} \quad \text{if } \pm x \geqslant 0, \tag{1}$$

where the dot represents derivatives with respect to the time variable and such derivatives for the second variable have been normalized in both sides.

Our main result is as follows.

Theorem. For any natural number *n,* system (1) is, in a neighbourhood of the origin, topologically equivalent to a systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y + \displaystyle\sum_{k=1}^{n} \mu_{2k}^{\pm} y^{2k} \\ \pm 1 \end{pmatrix} + \sum_{k \geqslant 2n} \mathbf{G}_k^{\pm}(\mathbf{x}), \tag{2}$$

where the symbol '+' applies for the right side and the symbol '-' does for the left side, and the final terms $\mathbf{G}_k$ are quasi-homogeneous polynomial vector fields of type (2,1) and degree *k*, in the terminology of the theory of quasi-homogeneous vector fields.

The coefficients of the even powers in the variable *y* are obtained in an algorithmic way in terms of the coefficients of system (1). Thanks to the above theorem, it is possible to compute in a straightforward way the coefficients of the half-return maps. We take advantage of the fact that such half-return maps are analytical involutions at the origin. We apply such procedure to a rather general family of linear-quadratic systems emphasizing the maximum number of limit cycles than can bifurcate from the origin.

3. Concluding Remarks

Specific normal forms are proposed for non-smooth systems with a pseudo-focus point, looking for facilitating the computations of the half-return maps, getting also an easy characterization of the order for the pseudo-focus. As a corollary, the maximal number of bifurcating limit cycles from perturbations without introducing sliding sets is obtained.

As a relevant application, we start the study of linear-quadratic systems with a pseudo-focus, which even not completely finished has already given rise to a rich variety of behaviors.

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