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# On Fractional Viscosity and Material Instability

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**Abstract:** Constitutive equations of materials are the key elements in material instability problems. Such equation may include fractional derivatives to describe well known material behaviours as creep and relaxation. Stability investigation can be performed as usually in the theory of dynamical systems. For this reason a dynamical system should be formed from the basic equations describing the motion of a solid body. This system of equations consists of the equations of motion, (Cauchy's first and second equations) the kinematic equation and the constitutive equation itself. A state of the material can be identified as a steady state solution of that dynamical system. Then periodic perturbations are added to that solution and stability and bifurcation analysis of it leads to conditions on the material constants of the constitutive equation. Stability and bifurcation behaviour should exhibit generic nature to have a physically acceptable mathematical description. The main question to answer is: how the presence of fractional derivatives in constitutive equations effects the possible forms of constitutive equations.

**Keywords:** fractional derivative, material instability, rate-dependence

## 1. Introduction

There are several instability phenomena appearing in material tests like flutter or shear-banding and they should be recognised as solutions of the system of equations describing the motion of a material body. Stability analysis is quite obvious by using dynamical systems theory and it presents a clear classification for the types of unstable behaviours. Unfortunately, such classification is impossible for rate-independent materials. Rate effects could be derived from the presence of viscosity. In a physical point of view, viscosity is deduced from material memory effects. Such way was followed by Rabotnov [1] in studying creep and relaxation phenomena of solid mechanics. By taking experimental results into account, his approach leads to hereditary mechanics with fractional order integral operators. Since then, several studies have dealt with the connection between hereditary approach and rate dependence and proved the equivalence [2], [3], when ”fractional order rate” is used. Thus Bagley's viscoelastic material model [4] is a direct consequence of the unity of creep and relaxation phenomena and of experimental data.

## 2. Stability and Bifurcation Analysis

The classical description of continuum mechanics consists of three groups of equations, such as Cauchy's first and second equations of motion, the kinematic equation and the constitutive equation. The simplest possible case is a uniaxial problem with small deformations. Then such equations are

$$\rho\dot{v} = \frac{\partial \sigma}{\partial x}, \quad \dot{\varepsilon} = \frac{\partial v}{\partial x} \tag{1}$$

and the constitutive equation $F(\varepsilon, \sigma, \dot{\varepsilon}, \dot{\sigma}, \dots) = 0$. When Bagley's model is used, the constitutive equation could be in form

$$b_0 \sigma = a_0 \varepsilon + a_1 D^{\alpha} \varepsilon, \tag{2}$$

where

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(\xi)(t-\xi)^{-\alpha} d\xi$$

denotes Riemann-Liouville's derivative [5]. Then equations (1), (2) and (3) should be transformed into velocity field v and the stability of the steady state solution $v_0$ of dynamical system

$$b_0 \rho \ddot{v} - (a_0 + a_1 D^{\alpha}) \frac{\partial^2 v}{\partial x^2} = 0 \tag{3}$$

should be studied. By using harmonic perturbations

$$\tilde{v} = \tilde{v}_0 v_t(t) \exp(\omega x)$$

of $v_0$, solutions $\lambda_i$ of the characteristic equation of (3) should be calculated.

The locations of $\lambda_i$ in the complex plane decide on stability. State $v_0$ of the material is stable, when all eigenvalues $\lambda_i$ have arguments less then $\pm \frac{\pi\alpha}{2}$. By changing the load on the specimen, material parameters $a_0, a_1, b_0$ may be varied and its effect on $\lambda_i$ may cause loss of stability. It could be either a static of a dynamic bifurcation. In such a way, conditions can be found for material parameters. Such conditions include the order of the derivative α as an additional material constant.

When $a_1 \equiv 0$ in constitutive equation (3), a co-existent static and dynamic bifurcation should happen, which is highly non-generic. Such model is inappropriate for material instability analysis. Otherwise, regions of stable and unstable behaviours, static and dynamic bifurcation conditions can be identified in the space of material parameters.

## 3. Concluding Remarks

Fractional derivatives appear in the constitutive equations as a direct consequence of creep and relaxation phenomena. Material instability can be investigated by defining a fractional order dynamical system from the system of basic equations of solids.

## References


1. RABOTNOV YU, N: Equilibrium of an Elastic Medium with After-effect (in Russian). *Prikladnaya Matematika i Mekhanika (J. Appl. Math. Mech.)* 1948, **12**:53-62.
2. ROSSIKHIN Y. A, SHITIKOVA M. V: Application of Fractional Calculus for Dynamic Problems of Solid Mechanics: Novel Trends and Recent Results. *ASME Appl Mech Rev.* 2010, **63**:010801
3. KOELLER R. C: Applications of Fractional Calculus to the Theory of Viscoelasticity. *Journal of Applied Mechanics* 1984, 5**1**(2):299-307.
4. BAGLEY R. L: *Applications of Generalized Derivatives to Viscoelasticity.* .Dissertation, USA Air Force Institute of Technology, 1979.
5. PODLUBNY I: *Fractional Differential Equations.* Academic Press: San Diego, 1999.
6. RABOTNOV YU, N: Equilibrium of an Elastic Medium with After-effect (in Russian). *Prikladnaya Matematika i Mekhanika (J. Appl. Math. Mech.)* 1948, **12**:53-62.