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# Lyapunov Functions by interpolating numerical quadratures: Proof of Convergence

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**Abstract:** We consider methods to compute Lyapunov functions for nonlinear systems. We prove that the combination of a fast, but non-rigorous method, with a slow, but rigorous, method results in a fast and rigorous method. Further, we show that our combined always succeeds in generating a Lyapunov functions for a system with an exponentially stable equilibrium.

**Keywords:** Lyapunov function, numerical method, nonlinear system

## 1. Introduction

Lyapunov functions are an essential tool in the study of the qualitative behaviour of dynamical systems. They give information on attractors, repellers and basins of attraction without the knowledge of the solution to a system, whose dynamics are given by an ODE or an iteration; in particular they can be used to assert the stability of an equilibrium and give a rigid lower bound on its basin of attraction. The analytic computation of a Lyapunov function for a nonlinear system is a very hard problem and in general not feasible. Therefore, numerous numerical methods have been suggested. In the CPA method continuous and piece-wise affine Lyapunov functions are parameterised using feasible solutions to a linear programming problem [1] and one can show that this method always succeeds in generating a Lyapunov function for a system with an exponentially stable equilibrium [2]. Another approach is to generate values of the target Lyapunov function and then verify the constraints of the linear programming problem [3,4]. In the paper associated to this abstract we prove that this approach will always succeed in generate a Lyapunov function for a system with an exponentially stable equilibrium, i.e. we show the convergence of this technique.

## 2. Results and Discussion

We show that one can generate adequate values for a Lyapunov function using integral formulas in the case of time-continuous systems (ODEs) and summation formulas in the case of time-discrete systems. These values will fulfil the constraints of the linear programming problem and can therefore be interpolated over the whole domain to deliver a true Lyapunov function for the system. As an example, see Figure 1, where this methodology was used to generate a Lyapunov function for the van der Pol system. Note that it is orders of magnitude more efficient to generate the values using an integral- or summation formula and verify the constraints, than to solve the linear programming problem.

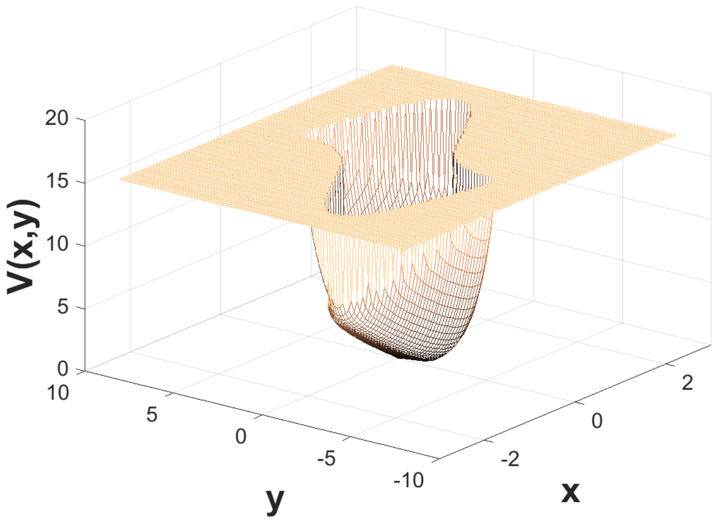


**Fig. 1.** Lyapunov function for the van der Pol system

## 3. Concluding Remarks

We lay the theoretical foundation for combining two methods for the generation of Lyapunov functions for nonlinear systems, either given by an ODE (continuous-time) or an iteration (discrete-time). The combined method inherits the numerical efficiency of the fast, but non-rigorous method, that evaluates numerically quadratures of numerically integrated solutions (continuous-time) or sums (discrete-time) and the rigorousness of the slower method solving linear programming problems.

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